Examiners' Report: Preliminary Examination in Mathematics Trinity Term 2021

January 28, 2022

Part I

A. STATISTICS

- **Numbers and percentages in each class.**
  See Table 1. Overall 196 candidates were classified.

Table 1: Numbers in each class (Preliminary Examination)

	Numbers					Percentages %				
	2021	(2019)	(2018)	(2017)	(2016)	2021	(2019)	(2018)	(2017)	(2016)
Distinction	60	(54)	(58)	(62)	(59)	30.61	(29.19)	(29.44)	(30.85)	(30.89)
Pass	124	(120)	(126)	(124)	(119)	63.27	(64.86)	(63.96)	(61.69)	(62.3)
Partial Pass	7	(8)	(10)	(13)	(7)	3.57	(4.32)	(5.08)	(6.47)	(3.66)
Incomplete	2	(1)	(0)	(0)	(0)	1.02	(0.54)	(0)	(0)	(0)
Fail	3	(2)	(3)	(2)	(6)	1.53	(1.08)	(1.52)	(0.99)	(3.14)
Total	196	(185)	(197)	(201)	(191)	100	(100)	(100)	(100)	(100)

- **Numbers of vivas and effects of vivas on classes of result.**
  As in previous years there were no vivas conducted for the Preliminary Examination in Mathematics.
- **Marking of scripts.**
  As in previous years, no scripts were multiply marked by Moderators; however all marking was conducted according to a detailed marking scheme, strictly adhered to. For details of the extensive checking process, see Part II, Section A.

B. New examining methods and procedure in the 2021 examinations

In light of the ongoing COVID-19 pandemic, the University changed the examinations to an open-book format and rolled out a new online examinations platform. An additional 30 minutes was added on to the exam duration to allow candidate the technical time to download and submit their examination papers via Inspera.

C. Changes in examining methods and procedures currently under discussion or contemplated for the future

The department intends to hold in person exams in Trinity Term 2022.

D. Notice of examination conventions for candidates

The Notice to Candidates, containing details of the examinations and assessment, including the Examination Conventions, was issued to all candidates at the beginning of Trinity term. All notices and the Examination Conventions in full are available at

https://www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/examination-conventions.

Part II

A. General Comments on the Examination

Acknowledgements

The Moderators are extremely grateful to the academic administration team, and in particular to Elle Styler and Charlotte Turner-Smith, for their hard work in running the examinations system and in supporting the Moderators throughout the year.

We also thank Waldemar Schlackow for maintaining and running the examination database and in particular for his assistance during the final examination board meeting.

We express our sincere thanks to Dr Estelle Massart for administering the Computational Mathematics projects. We would also like to thank the Assessors Dr Maria Christodoulou, Dr Beth Romano, Dr Kyle Pratt, Dr Lucy Auton, and Dr Francis Bishoff for their assistance with marking.

Timetable

The examinations began on Monday 21st June and ended on Friday 25th June.

Factors Affecting Performance

A subset of the Moderators attended a pre-board meeting to band the seriousness of circumstances for each application of factors affecting performance received from the Proctors' office. The outcome of this meeting was relayed to the Moderators at the final exam board. The moderators gave careful regard to each case, scrutinised the relevant candidates' marks and agreed actions as appropriate.

Mitigating Circumstance Notice to Examiners

A subset of the examiners (the 'Mitigating Circumstances Panel') attended a pre-board meeting to band the seriousness of the individual notices to examiners. The outcome of this meeting was relayed to the Examiners at the final exam board, who gave careful regard to each case, scrutinised the relevant candidates' marks and agreed actions as appropriate.

The full board of examiners considered 99 notices in the final meeting. The examiners considered each application alongside the candidate's marks. The outcomes for these have been recorded on a spreadsheet report on Mitigating Circumstances Notice to Examiners from Part A. All candidates with certain conditions (such as dyslexia, dyspraxia, etc.) were given special consideration in the conditions and/or time allowed for their papers, as agreed by the Proctors. Each such paper was clearly labelled to assist the assessors and examiners

Setting and checking of papers and marks processing

The Moderators first set questions, a checker then checked the draft papers and, following any revisions, the Moderators met in Hilary term to consider the questions on each paper. They met a second time to consider the papers at the end of Hilary term making further changes as necessary before finalising the questions. A meeting was held in early Trinity term for a final proof read. The Camera Ready Copy (CRC) was prepared and each Moderator signed off the papers. The CRC was submitted to Examination Schools in week 4 of Trinity term.

Candidates accessed and downloaded their exam papers via the Inspera system at the designated exam time. Exam responses were uploaded to Inspera and made available to the Exam Board Administrator 25-33.5 hours after the exam paper had finished via One Drive.

The process for Marking, marks processing and checking was adjusted accordingly to fit in with the online exam responses. Assessors had a week to return the marks on the mark sheets provided. A check-sum was also carried out to ensure that marks entered into the database were correctly read and transposed from the mark sheets.

All scripts and completed mark sheets were returned, if not by the agreed due dates, then at least in time for the script-checking process.

A team of graduate checkers, under the supervision of Elle Styler, reviewed the mark sheets for each paper of this examination, carefully cross checking against the mark scheme to spot any unmarked questions or parts of questions, addition errors or wrongly recorded marks. Also sub-totals for each part were checked against the mark scheme, noting correct addition. In this way a number of errors were corrected, each change was approved by one of the examiners who were present throughout the process.

Determination of University Standardised Marks

The candidates under consideration are Mathematics and Mathematics & Statistics candidates, 196 in total. We do not distinguish between them as they all take the same papers.

Marks for each individual paper are reported in university standardised form (USM) requiring at least 70 for a Distinction, 40–69 for a Pass, and below 40 for a Fail.

The Mathematics Teaching Committee issued each examination board with broad guidelines on the proportion of candidates that might be expected in each class. This was based on the average proportion in each class over the past five years (excluding 2020, when no Preliminary Examinations were held).

The raw marks were recalibrated to arrive at the USMs reported to candidates, adopting the procedures outlined below. These procedures are similar to the ones used in previous years.

To ensure equal weightings across all subjects, papers were first standardised to have broadly similar proportions of candidates attaining each class. A piecewise linear mapping was adopted to produce a USM from a raw mark. The default algorithm for each paper works as follows.

1. Candidates' raw marks for a given paper are ranked in descending order. Here the population data used is the set of marks for all candidates in Mathematics or Mathematics & Statistics.

2. The default percentages p_1 of Distinctions and p_2 of nominal upper seconds (USM 60-69) are selected, these percentages being similar to those adopted in previous years.

3. The candidate at the p_1 percentile from the top of the ranked list is identified and assigned a USM of 70. Let the corresponding raw mark be denoted by R_1.

4. Similarly, the candidate at the $(p_1 + p_2)$ percentile from the top of the list is assigned a USM of 60 and the corresponding raw mark is denoted by R_2.

5. The line segment between $(R_1, 70)$ and $(R_2, 60)$ is extended linearly to USMs of 72 and 57 respectively. Denote the raw marks corresponding to USMs of 72 and 57 by C_1 and C_2 respectively. For a graph of the mapping between raw marks and USMs, a line segment is drawn, connecting $(C_1, 72)$ to $(100, 100)$ with a further line segment between $(C_2, 57)$ and $(C_1, 72)$.

6. A line segment through $(C_2, 57)$ is extended down towards the vertical axis, as if it were to join the axis at $(0, 10)$, but the line segment is terminated at a USM of 37. The associated raw mark at the termination point is denoted C_3.

7. Finally a line segment between $(C_3, 37)$ and $(0, 0)$ completes the graph of the piecewise linear mapping between the raw marks and the USM.

Thereby a piecewise linear map is constructed whose vertices, at $\{(0, 0), (C_3, 37), (C_2, 57), (C_1, 72), (100, 100)\}$, are located away from any class boundaries.

A first run of the outlined scaling algorithm was performed. It was confirmed that the procedure resulted in a reasonable proportion of candidates in each class. The Moderators then used their academic judgement to make adjustments where necessary as described below. The Moderators were not constrained by the default scaling map and were able, for example, to insert more vertices if necessary.

To obtain the final classification, a report from each Assessor was considered, describing the apparent relative difficulty and the general standard of solutions for each question on each paper. This information was used to guide the setting of class borderlines on each paper.

The scripts of those candidates in the lowest part of each ranked list were scrutinised carefully to determine which attained the qualitative class descriptor for a pass on each paper. The gradient of the lower section of the scaling map was adjusted to place the pass/fail borderline accordingly.

Careful consideration was then given to the scripts of candidates at the Distinction/Pass boundary.

Adjustments were made to the scaling maps where necessary to ensure that the candidates' performances matched the published qualitative class descriptors.

The Computational Mathematics assessment was considered separately. In consultation with the relevant Assessor it was agreed that no recalibration was required, so the raw marks (out of 40) were simply multiplied by 2.5 to produce a USM.

Finally, the class list for the cohort was calculated using the individual paper USMs obtained as described above and the following rules:

Distinction: both $Av_1 \geq 70$ and $Av_2 \geq 70$ and a mark of at least 40 on each paper and for the practical assessment;

Pass: not meriting a Distinction and a USM of at least 40 on each paper and for the practical assessment;

Partial Pass: awarded to candidates who obtained a standardised mark of at least 40 on three or more of Papers I-V but did not meet the criteria for a pass or distinction;

Fail: a USM of less than 40 on three or more papers.

Here Av_2 is the average over the five written papers, weighted by length, and Av_1 is the weighted average over these papers together with Computational Mathematics (counted as one third of a paper). The Moderators verified that the overall numbers in each class were in line with previous years, as shown in Table 1.

The vertices of the final linear model used in each paper are listed in Table 2, where the x-coordinate is the raw mark and the y-coordinate the USM.

Table 2: Vertices of final piecewise linear model

Paper	Positions of vertices				
I	(0,0)	(33.66,37)	(58.64,57)	(85.6,72)	(100,100)
II	(0,0)	(28,39)	(43.7,57)	(75.2,72)	(100,100)
III	(0,0)	(32.97,37)	(57.4,57)	(90.4,72)	(120,100)
IV	(0,0)	(25.62,37)	(44.6,57)	(71.6,72)	(100,100)
V	(0,0)	(21,40)	(34.2,57)	(58.2,72)	(80,100)
CM	(0,0)	(33,86)			(40,100)

Table 3 gives the rank list of average USM scores, showing the number and percentage of candidates with USM greater than or equal to each value.

Table 3: Rank list of average USM scores

		Candidates with USM $\geq x$	
USM (x)	Rank	Number	%
94	1	1	0.51
86	2	3	1.53
85	4	4	2.04
83	5	5	2.55
82	6	6	3.06
81	7	8	4.08
80	9	11	5.61
79	12	12	6.12
78	13	21	10.71
77	22	27	13.78
76	28	29	14.8
75	30	33	16.84
74	34	36	18.37
73	37	40	20.41
72	41	43	21.94
71	44	51	26.02
70	52	58	29.59
69	59	62	31.63
69	59	62	31.63
68	63	72	36.73
67	73	82	41.84
66	83	97	49.49
66	83	97	49.49
66	83	97	49.49

Table 3: Rank list of average USM scores (continued)

		Candidates with USM $\geq x$	
USM (x)	Rank	Number	%
65	98	109	55.61
65	98	109	55.61
64	110	117	59.69
63	118	132	67.35
62	133	138	70.41
61	139	147	75
61	139	147	75
60	148	157	80.1
59	158	163	83.16
58	164	169	86.22
57	170	172	87.76
57	170	172	87.76
56	173	175	89.29
55	176	179	91.33
54	180	182	92.86
53	183	183	93.37
52	184	185	94.39
50	186	188	95.92
50	186	188	95.92
48	189	190	96.94
46	191	192	97.96
38	193	193	98.47
37	194	194	98.98
30	195	195	99.49
0	196	196	100

Recommendations for Next Year's Examiners and Teaching Committee

None.

B. Equal opportunities issues and breakdown of the results by gender

Table 4 shows the performances of candidates broken down by gender.

Table 4: Breakdown of results by gender

Class	Number								
	2021			2019			2018		
	Female	Male	Total	Female	Male	Total	Female	Male	Total
Distinction	7	53	60	8	46	54	7	51	58
Pass	50	74	124	49	71	120	57	69	126
Partial Pass	2	5	7	4	4	8	6	4	10
Incomplete	0	2	2	0	1	1	0	0	0
Fail	3	0	3	1	1	2	2	1	3
Total	62	134	197	62	123	185	72	125	197
Class	Percentage								
	2021			2019			2018		
	Female	Male	Total	Female	Male	Total	Female	Male	Total
Distinction	11.29	39.55	25.42	12.9	37.4	29.19	9.72	40.8	29.44
Pass	80.65	55.22	67.93	79.03	57.72	64.86	79.17	55.2	63.96
Partial Pass	3.23	3.73	3.48	6.45	3.25	4.32	8.33	3.2	5.08
Incomplete	1.49	0	0.74	0	0.81	0.54	0	0	0
Fail	4.84	0	2.42	1.61	0.81	1	2.78	0.8	1.52
Total	100	100	100	100	100	100	100	100	100

C. Statistics on candidates' performance in each part of the Examination

The number of candidates taking each paper is shown in Table 5. The performance statistics for each individual assessment are given in the tables below: Paper I in Table 6, Paper II in Table 7, Paper III in Table 8, Paper IV in Table 9, Paper V in Table 10 and Computational Mathematics in Table 11. The number of candidates who received a failing USM of less than 40 on each paper is given in Table 5.

Note that Paper I, II and IV are marked out of 100 (being 3 hours in duration), Paper III is marked out of 120 (being 3.5 hours in duration) and Paper V is marked out of 80 (being 2.5 hours in duration).

Table 5: Numbers taking each paper

Paper	Number of Candidates	Avg RAW	StDev RAW	Avg USM	StDev USM	Number failing
I	194	72.65	14.61	65.82	11.57	7
II	195	60.88	14.78	65.33	9.68	4
II	195	76.44	15.39	66.15	8.87	2
IV	195	60.14	14.67	65.71	10.78	5
V	195	47.55	3.01	65.65	11.75	4
CM	196	31.39	5.98	80.98	14.79	0

Table 6: Statistics for Paper I

Question Number	Average Mark All	Average Mark Used	Std Dev	No. of Attempts Used	No. of Attempts Unused
Q1	14.96	14.96	3.49	175	0
Q2	10.88	10.88	4.63	98	0
Q3	16.88	16.88	3.28	184	0
Q4	15.95	15.95	3.14	119	0
Q5	14.8	14.8	4.25	170	0
Q6	11.91	11.91	4.52	74	0
Q7	14.34	14.35	3.87	140	1

Table 7: Statistics for Paper II

Question	Average Mark		Std	No. of Attempts	
Number	All	Used	Dev	Used	Unused
Q1	15.34	15.34	3.37	161	0
Q2	13.57	13.57	4.43	128	0
Q3	10.5	10.5	4.47	100	0
Q4	11.31	11.31	2.04	185	0
Q5	8.95	8.95	5.14	77	0
Q6	15.52	15.52	4.16	126	0
Q7	9.88	9.88	4.44	190	0

Table 8: Statistics for Paper III

Question	Average Mark		Std	No. of Attempts	
Number	All	Used	Dev	Used	Unused
Q1	18.01	18.01	2.63	193	0
Q2	14.70	14.73	3.86	121	1
Q3	14.23	14.28	4.20	76	1
Q4	14.05	14.05	3.55	175	0
Q5	12.32	12.35	3.51	175	1
Q6	10.78	11	3.75	39	6
Q7	9.74	9.74	4.02	169	0
Q8	8.94	8.94	4.27	147	0
Q9	11.56	11.56	4.40	48	0

Table 9: Statistics for Paper IV

Question	Average Mark		Std	No. of Attempts	
Number	All	Used	Dev	Used	Unused
Q1	13.14	13.14	4.93	128	0
Q2	12.16	12.25	4.36	139	1
Q3	11.44	11.61	3.98	121	2
Q4	8.66	8.71	4.85	112	1
Q5	13.66	13.66	4.58	164	0
Q6	9.38	9.51	4.17	98	4
Q7	14.31	14.31	4.22	195	0

Table 10: Statistics for Paper V

Question Number	Average Mark All	Used	Std Dev	No. of Attempts Used	Unused
Q1	10.84	10.84	4.67	167	0
Q2	9.02	9.02	4.66	123	0
Q3	14.32	14.32	4.46	92	0
Q4	16.72	16.72	3.40	194	0
Q5	8.26	8.26	4.62	117	0
Q6	11.32	11.32	4.40	73	0

Table 11: Statistics for Computational Mathematics

Question Number	Average Mark All	Used	Std Dev	No. of Attempts Used	Unused
Project A	15.48	15.48	3.43	122	0
Project B	16.04	16.04	2.58	91	0
Project C	16.11	16.11	2.69	174	0

D. Comments on papers and on individual questions

Paper I

Question 1 popular question and reasonably well-done and there were various approaches to (a). Quite a few students completed (a)(i) and (ii) but struggled to complete (iii), claiming (falsely) that rowspace(AB) = rowspace(A) if B is invertible (or equivalent). Their dimensions are equal, but they are not equal as spaces.

(b)(i) is equivalent to showing that M is diagonalizable. Only partial credit was given to those scripts which immediately stated the characteristic polynomial or eigenvalues, as calculated by some software, without further explanation. As the eigenvalues are distinct then M is diagonalizable and there is no need to explicitly calculate P such $D = P^{-1}MP$: Only diagonal matrices Δ commute with D and so the matrices that commute with $M = PDP^{-1}$ are of the form $P\Delta P^{-1}$. Many followed this intended line of thinking. However the setter missed the much easier solution to (b)(iii) noting that scalar multiples of I_3 or of M provide infinitely many matrices that commute with M.

Question 2 A less popular question and not as well done. In (a)(ii) the linear map of use is $F^n \to F$ given by $x \mapsto a \cdot x$. In (a)(iv) the kernel of T consists of the antisymmetric matrices and the image of the symmetric matrices. In (b) many scripts claimed the dimension of V_n to be n when it is $n + 1$. To answer (b)(iii) quite a few scripts sought to quote Bézout Lemma but this received no credit unless a proof was provided. Rather the map $T : V_{n-1} \times V_{n-1} \to V_{2n-1}$ given by $(u, v) \mapsto up + vq$ is a linear map between spaces of equal dimension. By (b)(ii) the kernel of T is $(0, 0)$ and so T is injective and hence surjective. In particular 1 is in the image of T.

Question 3 (a)(i) Some did not check whether nonempty. (ii) Many attempted proofs by contradiction. (iii) Some attempted directly to show equality, without recourse to double inclusion. Mostly done well. (b) Some confused injective and surjective, mostly done well. (c) (i) When attempted, most were successful. (ii) Some claimed without proof that E is a basis.

Question 4 (a) (i) Some did not show the inequality is strict. (ii) Many referred to part (a) without considering case in which vectors are linearly dependent. (b) (i) Nearly all were successful. (ii) Some failed to note an eigenvector has nonzero norm, in which case it can be divided out. (iii) Many invoked the spectral theorem, though some made attempts with characteristic polynomials. (iv) Mostly done well.

Question 5 (a)(i) Some failed to comment on all of the axioms of a group. (ii) Some assumed X was finite or equal to a set of integers. (iii) Some did not appreciate that $\theta(g)$ is a function. (b) (i) and (ii) some assumed the right inverse given was also a left inverse rather than proving this and similarly with the identity. (iii) Some did not show the identity is two-sided or refer to two-sided inverses. (c) (i) Some tried to proceed without reference to the fact X is assumed finite. (ii) Mostly well done when attempted.

Question 6 (a)(i) Partitions do not include the empty set. (ii) The sets are not assumed finite so cardinality may not be used. Some failed to show proposed bijections were well-defined. (iii) Mostly done well. (b) (i) Mostly done well. (ii) Seemed to be found very hard. Very few correct answers. (iii) Some failed to define the a group operation on the integers. (c) (i) Mostly done well. (ii) Mostly done well. (iii) Done by many even when other parts were not.

Question 7 (a) Well done. (b)(i) Some forgot to show A is a subgroup. (ii) Many referred to wikidots, but lost marks for not explaining the answer. (iii) Almost everyone did this well. (iv) Intended as an easy deduction from previous two parts though some went through the cases. (c) (i) Found to hardest. (ii) Mostly well done by those who got here.

Paper II

Question 1 contained items which were standard or close to questions in the exercise sheets.

In item a), many candidates correctly used the approximation property of the supremum/infimum or correctly used the 2-step definition and got the desired results. The common mistake was to only argue for upper bound and conclude it was the supremum without checking the minimality. The vast majority spotted a counter example to "$\sup(A \cdot B) = \sup(A) \cdot \sup(B)$".

In item b), most candidates managed to find the limit points by looking at residues modulo 4 but the major pitfall was to not argue the converse direction, by taking a general converging subsequence and conclude which were the possible limit points. Another common mistake was to treat infinity as a limit of a convergent subsequence.

Item c) required an idea similar to the one used in a question from the exercise sheet and most candidates managed to extract a bounded subsequence from the unbounded sequence $\tan(n)$. Lack of justification for the existence of a natural number in certain intervals or not stating the Bolzano-Weierstrass Theorem, as indicated in the question instructions, led to loss of points in this part.

Question 2 contained a mixture of standard questions and limits of new sequences.

In item a), the first half was typically okay, but in the second half, many candidates were unable to extract a subsequence converging to zero using the given condition. In this part, it was not allowed to assume that a Cauchy sequence is convergent (this needed to be proved, if required in the argument).

Item b) contained several limits to be considered and from the instructions, the students were only allowed to assume the results from algebra of limits. Vague sentences like “exponentials beat polynomials” or assuming $\lim_n \log(n)/n = 0$ without justification caused some loss of points. Manipulations using the “sandwich” theorem or the ratio test efficiently led to the correct answer.

The last item was new and those that attempted it managed to spot two subsequences converging to different limits. A very common mistake here was that some candidates evaluated $s(p) = p + 1$ for p prime (where $s(n)$ is the sum of all prime factors of n, counted with multiplicity), suggesting that some candidates treated 1 as a prime number.

Question 3. There were a wide range of marks for this question. Many candidates had correct definitions for (a), though mistakes with indices were fairly common. Candidates generally had the right idea for (b), though many did not provide the necessary level of rigor in their proofs. There were a lot of mistakes in part (c): in (i), many people showed that a certain subsequence of the partial sums converged, and thought that this was enough to show the series converged. In (ii) and (iii), it was fairly common for candidates to write rearrangements of the series that did not converge.

Question 4. This is by far the most popular question, it was attempted by almost all candidates. Part (a) is a straightforward bookwork that was well done by the majority of the candidates. The most difficult part is (b)(ii), a lot of candidates wrongly assumed that a continuous function must be monotone on some sub-intervals.

Question 5. This is the least popular question. Most of the candidates, who attempted it, made good progress in part (a), but very few attempted (b). Those who attempted part (b) usually made reasonable progress. Some candidates used facts about the Cantor staircase function that are true, but require justification.

Question 6. It is the second most popular question but it turned out to be the easiest one. The average raw mark is 15. The main difficulty was in part (a), where many candidates were not sure how to use Rolle’s theorem to prove inequality. Solutions that do not use Rolle’s theorem got partial credit.

Question 7. The answers to this question seemed a bit rushed, and many candidates didn’t complete the full problem. In part (a), it was fairly common to lose a mark for not fully defining notation. For (b), many candidates gave correct solutions for part (i) but then incorrectly applied the theorem from (a) in part (ii). Most candidates who attempted (c) had good solutions, but the majority of the candidates did not complete the problem.

Paper III

Question 1 was answered by the vast majority of candidates, and solutions were generally very good. In part (c) some candidates spent a lot of time working with an incorrect or inappropriate trial solution for the particular integral; a bit more time spent thinking about a suitable form in advance might have helped reduce the complexity of calculations needed.

Question 2. In Q2 there were some incorrect uses of the chain rule in part (a) and also the first part of part (b)(i). Whilst many candidates did state one or both of the conditions on f and g needed in (a), these were not always stated at their point of use. A significant minority of candidates did not know the correct formula for the area of the region in part (c), nor were they able to work it out.

Question 3. Most candidates who attempted Q3(a) interpreted it well and there were some excellent solutions here. However, only a few candidates justified why their extremum was a maximum. In Q3(b) students missed the $y = x$ symmetry that might have simplified things, and in particular several candidates struggled to prove that there is only one critical point in the required region.

Question 4. Part (a) gave data from the first successful Covid vaccine trial, and asked for the computation of the efficacy. Quite a few answers skipped this part entirely. The first subpart asked for the completely standard confidence interval computation for the probability in a binomial observation. The most significant challenge here was to recognise which were the relevant numbers: The question asked about the probability that a Covid case was vaccinated — for which the stated total number of trial participants 43,448 was irrelevant — but some answers instead examined the probability that a vaccinated participant contracted Covid. Partial marks were given for these answers, if otherwise correct. Another common error was to use the variance rather than the standard deviation in the calculation. Many answers for subpart (ii) were too vague to be awarded full marks, saying that the interval for p would be transformed into an interval for $\varnothing$, without saying how this would be done. Full marks could be obtained either by pointing out that the function was decreasing and that the inverse function would be applied to the endpoints (in reverse order) or by explicitly solving the two inequalities.

Question 5. Part (b) considered the MLE calculation for a uniform distribution on $[-0; 0]$, slightly modified from the uniform distribution on $[0; 0]$ with which the students would already have been familiar. The most common error in subpart (i) was to state that the MLE would be max xi rather than max $[\chi,\iota]$, perhaps because this was the answer in the earlier version. Sub- part (ii) required computation of the mean square error, which could be done either with the variance-bias formula or directly. In either case, the pdf of the maximum needed to be computed. Most did this well, but some who had neglected the modulus in stating the $MLE-$ and even some who hadn't — computed the pdf for max xi, or otherwise spread out the pdf for the estimator over $[-0; 0]$, which yielded somewhat more involved algebraic computations. The final subpart asked for the MSE of a different estimator, and most answers here were substantially or completely correct.

Question 6. This question was not as popular as anticipated, and most candidates found it on the challenging side. Part (a) was overall done well, however some candidates lost marks by not stating the law of total probability, or by not establishing that there were two different cases to consider. From Part (b) marks were lost primarily by candidates not showing their work and thought process, and simply stating results, as opposed to not understanding the setup. Section (iv) challenged some candidates who did not manage to successfully work out how E[Y] and E[X—J=1] connect. Part (c) proved the most challenging with very few candidates even attempting it. Very few used the hint and linked this part to the start of the problem.

Question 7. Part (c) was a question about the covariance structure of linearly trans- formed random variables. The first subpart asked for a demonstration of the basic bilinearity formula for covariance, which had been done on a problem sheet. Unfortunately, because it was so close to the problem sheet question quite a few students skipped key steps of the demonstration by saying "By problem 1 of sheet 6" or "Because covariance is bilinear". These received part marks if otherwise correct. The second subpart required them to continue to the matrix transformation of the covariance matrix — for which 1 mark out of 3 was allotted — and then combine this with understanding from linear algebra of the possible rank of an orthogonally transformed matrix. Very few answers actually referred to the rank of the original covariance matrix — which was required to get full marks — and many seemed to implicitly assume that the matrix had full rank and that all eigenvalues were positive. Thus, the answer $k \leq p$ was by far the most common answer (where this part was not left out altogether). Some also assumed that the transforming matrix B must also be invertible, thus obtaining the answer $k = p$.

Question 8. A very popular statistics question with mixed overall performance. Many candidates had run out of time by the time they attempted this question, with some low scores clearly reflecting this. From Part (a), explaining why the MLE may not be uniquely defined proved the most challenging aspect, with some intuitive reasonings receiving partial marks if they were argued well. From Part (b), marks were lost in section (i) by failing to state independence or by not having the right variance. Heteroscedasticity also proved slightly confusing in section (ii) with some giving the right reasons for the wrong answer. Section (iv) was overall done very well by those who attempted it. Section (v) was the most challenging as many did not attempt to actually calculate the bias, leading to the suspicion that this reflected time management more than ability to work through this.

Question 9. Less popular than Question 8 but with a much stronger overall performance by those who attempted most parts. Again, many candidates had run out of time by the time they attempted this question, with some low scores clearly reflecting this. The overall standard of those who had time to complete it was very high. From Part (a), the calculation of standard error was the most challenging, with many conflating it with standard deviation. In section (iii) some marks were occasionally lost by careless wording on how widely applicable CLT is. From Part (b), some common mistakes on the graph were absence of axes labels and absence of PC5. Section (v) challenged many, who were not able to establish the correlation direction between PCs 2 and 3. The remainder of Part (b) was overall done very well.

Paper IV

Question 1. A popular question with many complete answers, though in (a)(ii) relatively few recognized and showed that P lies between A and B if and only if the angles CAB and CBA are acute. For (b) quite a few scripts showed the existence of the circumcentre (as a point equidistant from A, B, C) which did not explicitly answer the given question.

There were various approaches to most parts of the question, both successful and unsuccessful. The contrast between those answers that could set up an approach and notation clearly, argue directly and handle vector algebra and complex variables fluidly, and those that were inexplicit, logically meandering and needlessly introduced coordinates, was very marked.

Question 2. A popular question, often very well done. Many approached part (b) without the spectral theorem, choosing to rotate the conic to eliminate the mixed xy term. This is a valid approach, often gaining full credit, but is much lengthier than was intended. Unfortunately quite a few missed the last part of (a), asking for the area of the ellipse in terms of α, β, γ, presumably because this extra task was not highlighted well enough in the question's layout.

The projection C' has equation

$$x^2 + xy + y^2 + x + y = 1.$$

A translation of this curve gives one of the form studied (a), but many chose to approach the task afresh without making use of (a). The original curve C bounds a region which is $\sqrt{3}$ times bigger than the area bounded by C'. This was variously appreciated using elementary geometry, by determining the pre-image of the unit square in Π or by using the surface area formula for a graph.

Question 3. Parts (a) and (b) were largely well done. In (b)(iii) it needed to be appreciated that the rotating $(3, 4, 1)$ and $(12, 5, 7)$ into the xz-plane gives $(5, 0, 1)$ and $(13, 0, 7)$ and the formula in (ii) needs applying to these points.

(c)(i) was well done by many. It follows as

$$|(A\mathbf{r})_u \wedge (A\mathbf{r})_v| = |A\mathbf{r}_u \wedge A\mathbf{r}_v| = |\pm A(\mathbf{r}_u \wedge \mathbf{r}_v)| = |(\mathbf{r}_u \wedge \mathbf{r}_v)|$$

when A is orthogonal. A number of attempts sought to use the identity

$$|A\mathbf{u} \wedge A\mathbf{v}| = |\det A|\,|\mathbf{u} \wedge \mathbf{v}|$$

but this is not true in three dimensions (even though it is reminscent of the fact that in two dimensions $|\det A|$ is the area scaling factor). Many scripts proved the result that isometries between surfaces preserve area, which gained full credit, but was a harder, more general result than the one sought.

Part (c)(ii) was difficult. By a standard continuity argument it follows that

$$|A\mathbf{u} \wedge A\mathbf{v}| = |\mathbf{u} \wedge \mathbf{v}|$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$. Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ denote the columns of A. By choosing

$\mathbf{u} = \mathbf{i} + \lambda\mathbf{j}$ and $\mathbf{v} = \mathbf{k}$ (and similar) it follows that $\mathbf{c}_1 \wedge \mathbf{c}_2$, $\mathbf{c}_2 \wedge \mathbf{c}_3, \mathbf{c}_1 \wedge \mathbf{c}_3$ are orthonormal. Hence $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ are also orthonormal or equivalently A is orthogonal.

Question 4. The first part of 4(a) – deriving the expression given on the paper for $d\mathbf{L}/dt$ – was done correctly by nearly all students doing the questions. Many students also found the expressions for $\dot{h}$ and $\dot{E}$ but several struggled, in particular those who did not exploit the expression for $d\mathbf{L}/dt$ when considering $\dot{h}$ or use vector algebra for $\dot{E}$ and got lost in the algebra for more tedious approaches. The final part of (a) was done well in part by many students but not everyone gave more than one case for when E and h are conserved.

4(b) was attempted successfully by fewer candidates than (a). Some started well but gave incomplete answers for F_c, that did not have the proper r dependence. Another common error was to get the sign wrong for F_c. Those who attempted (b)(ii) and followed the hint got sensible results, but many candidates did not understand how to employ the hint appropriately.

Question 5. This was the most popular question. Part (a) was successfully done by almost all candidates that accepted it, with a minority failing to correctly solve the first order ODEs for $h(t)$ and θ. Part (b) was almost bookwork, and the major errors were failing to state clearly that and why h and energy (where it was used as starting point for the derivation) are conserved. Part (c) was the most challenging part, with quite a few candidates not even attempting this part. However, many candidates who did attempt the question got the first half of (c), that is, the derivations leading to (1) correctly. A subset of these then attempted the last part of the question and also found alternative to the one presented in the model solution.

Question 6. This question was of a similar popularity as question 4. Part (a) was done well by almost all candidates who attempted question 6. Part (b) was also done well, with occasional flaws in the argumentation or failing to complete the algebra to obtain the correct K (and J). Part (c) was the part most students struggled with. One difficulty was to get the stability analysis right, or even attempting it. The next hurdle was to get the equilibrium condition right, work out all equilibria. Only few candidates got the results for all equilibria.

Question 7. Part (a) was answered well by most candidates, although many solutions missed the finer points needed for full marks. There were some spurious methods introduced in part (b) where candidates did not use the correct version of Newton's method. In part (c) again some candidates omitted full details, for example did not note the need to update the Jacobian.

Paper V

Question 1. (a) Most candidates identified correctly the need for parabolic coordinates, but many had difficulties writing the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ in terms of u, v.

(b) Many candidates tried to calculate the work directly, by parametrising the edges of the trapezium and computing four line integrals. This led to many integration and arithmetic errors. A much faster approach was to use Green's Theorem as in the hint.

(c) There were multiple mistakes in the solutions in the parametrisation of the surface or the computation of the surface element. Also, several candidates invoked an erroneous argument involving symmetries.

Question 2. The most difficult part of this question turned out to be (b). In (b)(i), most candidates found it difficult to choose an appropriate $\mathbf{F}$ to simplify the computations. A common choice was $\mathbf{F} = \frac{1}{3}\mathbf{r}$, which leads to a correct, but complicated solution. A much simpler approach is to choose $\mathbf{F} = z\mathbf{k}$.

Another difficulty (but less common) was finding an appropriate v in (a)(iii) in order to apply (ii).

Question 3. The most common mistakes in the solutions had to do with the calculation of $d\mathbf{S}$, or choosing appropriate parametrisations in (a)(i). In (a)(ii), attention had to be paid to the orientation on the three boundary pieces.

Question 4. This question was generally well done, and most of the mistakes arose from errors in computation rather than conceptual difficulties. The most common mistakes were computational errors in determining the Fourier coefficients in part b). The common mistakes in part a) consisted in providing incomplete information, and in graphing the Fourier series instead of the function. The errors in part c) often arose from students being mislead by earlier mistakes in the question.

Question 5. Q5(a): There were some very good answers to 5(a) but the majority of attempts were poor and showed a serious lack of understanding of the method of characteristics. The allocated marks for this part of the question relied on correct answers and thus partial attempts scored very low. Not that many candidates attempted to sketch the profiles which accounted for over a third of the marks of part a.

The vast majority of candidates managed b(i) with ease although some candidates did struggle with this introductory calculus transformation. b(ii) the majority of candidates did notice that the coefficients of $y_{\eta\eta}, y_{\xi\xi}$ need to vanish. A lot of candidate missed that they must also ensure that the coefficient of $y_{\eta\xi} \neq 0$. At this point several candidates were stuck and couldn't see how to proceed to get a condition $H(B) > C$. This should be done by determining α and β in terms of B and C and ensuring that these exist, are real and distinct. One must then check that if α and β exist, are real and distinct that the coefficient of $y_{\eta\xi} \neq 0$, which was again often missed. Most candidates correctly calculated the general solution.

Question 6. Note that, there was a small error on the paper for Q6a. One candidate noted this.

The last bit of Q6a reads:

> Show that, when $a = 0$, then $T(x,t)$ is an even function of x.

it should instead read:

> Show that, when $a = 0$ and $b(x,t)$ is an even function of x, then $T(x,t)$ is also an even function of x.

In general showing uniqueness in part (a) was quite well done, but showing T was even was either not attempted and for those who did attempt this part the vast majority of the candidates missed the point. Those that did attempt this mostly showed that the PDE was invariant to the mapping $x \to -x$ but failed to show that the whole IBVP was invariant, which is crucial. This argument can be further strengthened by using uniqueness in that if we have 2 solutions to the same IBVP $T(x,t)$ and $\mathcal{T}(-x,t)$ these must be equal and thus the only solutions are even.

It appeared as though a lot of candidates ran out of time during 6(b). For those that did not this was generally well done. The questions asks for one to obtain 'the solution' – not 'a solution'. Thus uniqueness must be argued. There are 2 approaches to this: firstly, consider all the uncountably infinite number of constants and formally apply the principle of linear superposition to determine the general series solution; V is then unique and thus T. Alternatively, don't consider all the values for the constant but explain why 2 of the cases for this constant can be neglected, determine the solution for T and then use Q6(a) to argue uniqueness. Another point that was missed universally is that during the derivation of the Fourier series coefficient expression assumptions are made and these must be acknowledged (*i.e.,* the interchange of integration and summation).

E. Names of members of the Board of Examiners

- **Examiners: Prof. Dan Ciubotaru (Chair), Dr Richard Earl, Prof. Andreas Muench, Prof. Dmitry Belyaev, Prof. Tom Sanders, Prof. David Steinsaltz, Dr Cath Wilkins.**
- **Assessors: Dr Maria Christodoulou, Dr Beth Romano, Dr Kyle Pratt, Dr Lucy Auton, Dr Francis Bishoff.**